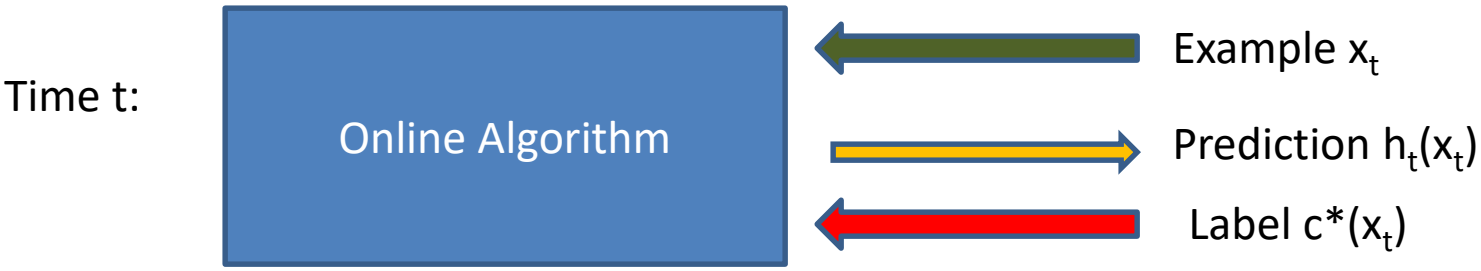
Online Algorithms

Outline

- Online Model: Mistake Bound
  - Simple algorithms
    - CON, HAL, ELIM
    - Online to PAC
- Linear Separator
  - Percpetron
    - Realizable case
    - Unrealizable case – Hinge loss
  - Winnow

Online Model

- Examples arrive sequentially
- Need to make a prediction
  - Afterwards observe the outcome
- No distributional assumptions
- Goal: Minimize the number of mistakes



Con Algorithm

- Realizable setting.
- Modest goal:
  - At most |C|-1 mistakes
- How?

- At time t
  - Let C_t the consistent concepts on $x_1, x_2 .. x_{t-1}$
    $\forall c \in C_t \;\; \forall i \leq t-1$
    $c(x_i) = c^*(x_i)$
  - Let $h_t \in C_t$
    - arbitrary
  - Predict $h_t(x_t)$
  - Observe $c^*(x_t)$

CON Algorithm

- **Theorem**: For any concept class C, CON makes the most $|C| - 1$ mistakes
- **Proof**: Initially $C_1 = C$.
- After each mistake$|C_t|$ decreases by at least 1
- $|C_t| \geq 1$, since $c^* \in C$ at any t
- Therefore number of mistakes is bound by

$$|C| - 1$$

Can we do better ?!

HAL – halving algorithm

- Change selection of h_t
- At time t, given x_t
  - For every $c \in C_t$
  - Compute $c(x_t)$
  - Predict the majority
    - $h_t(x) = M\{c(x): c \in C_t\}$
      - M is the majority
    - Might be that $h_t \notin C_t$

- HAL algorithm
- At time t
  - Let C_t the consistent concepts on $x_1, x_2 .. x_{t-1}$
    $\forall c \in C_t \forall i \leq t-1$
    $c(x_i) = c^*(x_i)$
  - Let $h_t = M\{c(x): c \in C_t\}$
  - Predict $h_t(x_t)$
  - Observe $c^*(x_t)$

HAL –halving algorithm

- **Theorem**: For any concept class C, HAL makes the most $\lfloor \log_2 |C| \rfloor$ mistakes
- **Proof:** Initially $C_1 = C$.
- After each mistake $|C_{t+1}| \leq \frac{1}{2}|C_t|$
  - majority of consistent concepts were wrong.
- Therefore number of mistakes is bound by $\lfloor \log_2 |C| \rfloor$

Example: Learning OR of literals

- Inputs: (x_1, … , x_n)
- Literals : z_i , $\bar{z}_i$
- OR functions:
  - $z_1 \vee \bar{z}_4 \vee z_7$
- Realizable case:
  - c*(z) is an OR
    - L^* are the literals
- ELIM algorithm:
  - Initialize:
    $L_1 = \{z_1, \bar{z}_1, \dots, z_n, \bar{z}_n\}$
  - Time t, receive
    $x_t = (x_1, \dots, x_n)$
  - Predict $OR(L_t, x_t)$
  - Receive $c^*(x_t)$
    - If $c^*(x_t)$ negative
      - delete from L_t any positive literals of z.

What is the MAXIMUM number of mistakes?

Analysis of ELIM

- Properties:
  - $L^* \subset L_1$
  - $L_t \subseteq L_{t-1}$
  - $L^* \subseteq L_t$
  - On mistake
    - $|L_t| \leq |L_{t-1}| - 1$
  - First mistake:
    - $|L_{t-1}| - |L_t| = n$
  - Later mistakes
    - $|L_{t-1}| - |L_t| \geq 1$

- No mistake when $c^*(x_t)=1$
  - $L^* \subseteq L_t$
- Number of mistakes
  - Initially $|L_1| = 2n$
  - First mistake:
    - Reduce by *n*
  - Each additional mistake
    - Reduce by at least 1
  - Maximum number of mistakes *n+1*

Regret Minimization

- What about non-realizable setting?
- We can guarantee:
  - Sequence of length T; average loss:
  - $Loss(online) \leq \min_{c \in C} Loss(c) + \sqrt{\frac{\log |C|}{T}}$
  - Best expert problem.
    - $Regret = Loss(online) - \min_{c \in C} Loss(c)$
- Not in the course !

Mistake Bound and PAC

- Are they equivalent
  - Class C is PAC learnable iff
  - Class C has finite Mistake Bound
- NO!
  - Prefix on of [0,1]
  - PAC learnable
  - No finite mistake bound
    - Why?
- What about the other direction
  - Algorithm A learns C in mistake bound model
  - Does A learn C in PAC model?
- We will show A_{PAC} that PAC learns C
  - A_{PAC} uses A

Conservative Online Algorithm

- A is conservative if
  - for every sample x_t if $c^*(x_t) = h_t(x_t)$ then
    - $h_{t+1} = h_t$
- Conservative algorithm
  - Changes hypothesis only after mistakes
  - After mistakes always changes hypothesis.
- Goal: show how to convert any algorithm A to a conservative algorithm A'
  - Same mistake bound
- Basic idea:
  - Feed A only mistakes

Conservative Mistake Bound Algorithm

- Given A Define A'
  - Initially $h_0 = A(\emptyset)$
  - At time t:
    - Predict $h_t(x_t)$
    - IF $c^*(x_t) = h_t(x_t)$,
      - THEN $h_{t+1} = h_t$
      - ELSE
        - » $ER = ER \,||x_t$
        - » $h_{t+1} = A(ER)$

- For any input sequence
  - If A' makes M mistakes
  - Then A makes M mistakes on ER

- Claim: Mistake Bound of A and A' identical

Building A_{PAC}

- Given algorithm A
  - At most M mistakes
  - A is conservative
- A_{PAC} works in stages
  - At most M+1
- Stage i:
  - Run algorithm A
  - If makes error
    - Start stage i+1
  - No error for $m_i = \frac{1}{\epsilon}\log\frac{1}{\delta_i}$
    - Output h_i

- Termination
  - Each stage A makes an error
  - At most M completed stages
- Performance
  - If h_i is $\epsilon - bad$ prob output:
  - $(1-\epsilon)^{m_i} \leq e^{-\epsilon m_i} = \delta_i$
- Sample size
  - $\sum_{i=1}^{M+1} m_i \leq \frac{M}{\epsilon}\left(\frac{M+1}{2} + \log\frac{1}{\delta}\right)$
- Confidence
  - Set: $\delta_i = \frac{\delta}{2^i}$
    - $\sum_i \delta_i \leq \delta$

Learning Linear Separators

- Input $\{0,1\}^d$ or R^d
- Linear Separator
  - weights w in R^d and threshold θ
  - hypothesis $h(x)=+1$ iff

$$\langle w,x \rangle = \Sigma\, w_i\, x_i \geq \theta$$

- Simplifying assumptions:
  - $\theta=0$ (add coordinate x_0 such that $x_0=1$ always)
  - $||\, x||=1$

Perceptron - Algorithm

- Initialize $w_1=(0,\ldots,0)$
- Given example x_t,
  - predict positive iff $\langle w_t, x_t \rangle \geq 0$
- On a Mistake t: $w_{t+1} = w_t + c_t(x)\, x_t$,
  - Mistake on negative (i.e., $c^*(x)=+1$): $w_{t+1} = w_t + x_t$.
  - Mistake on positive (i.e., $c^*(x)=-1$): $w_{t+1} = w_t - x_t$.

Perceptron - motivation

- **False Negative**
  - $c_t(x) = +1$
  - $\langle w_t, x_t \rangle$ negative
  - after update

$\langle w_{t+1}, x_t \rangle$

$= \langle w_t, x_t \rangle + \langle x_t, x_t \rangle$

$= \langle w_t, x_t \rangle + 1$

- **False Positive**
  - $c_t(x) = -1$
  - $\langle w_t, x_t \rangle$ positive
  - after update

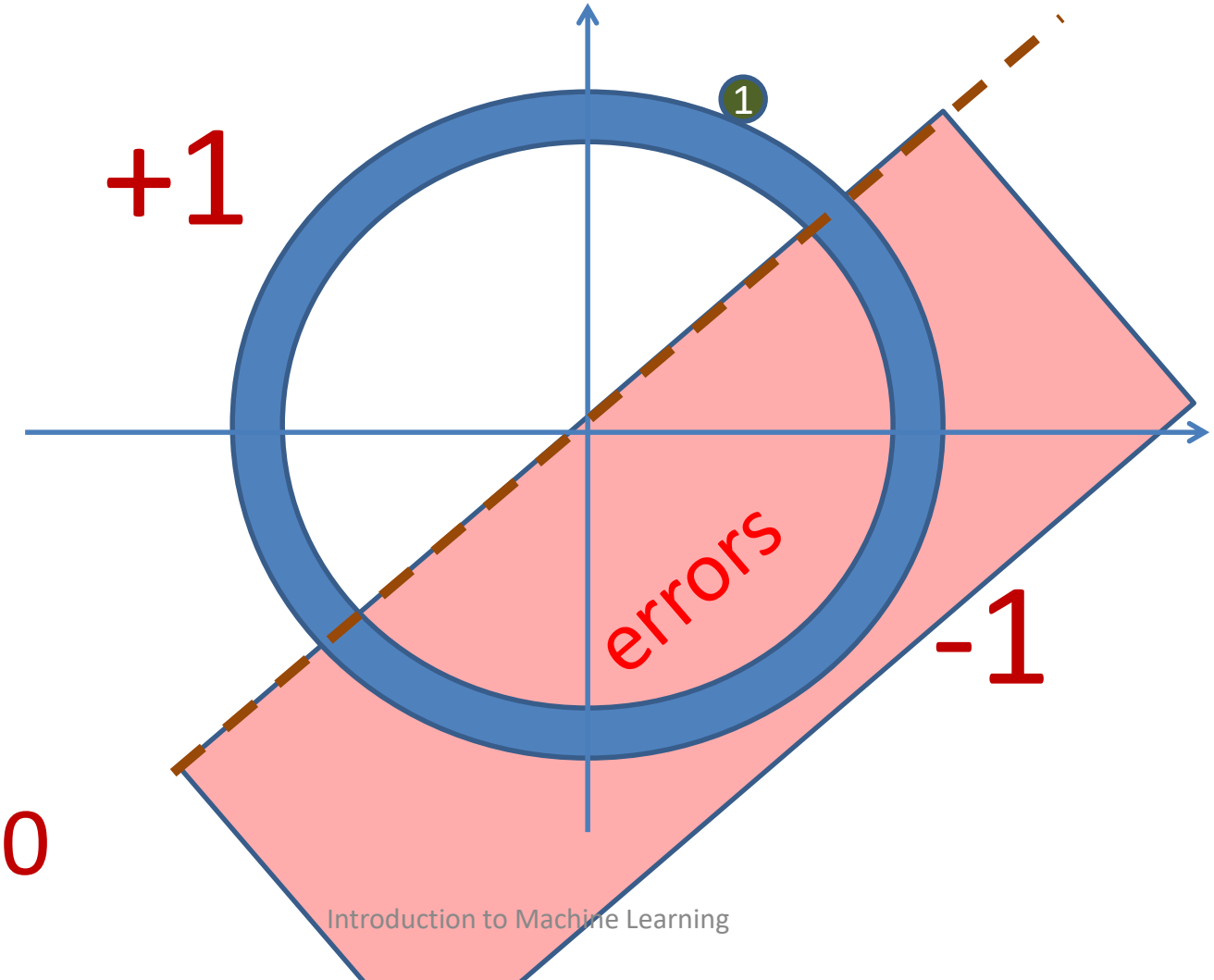
$\langle w_{t+1}, x_t \rangle$

$= \langle w_t, x_t \rangle - \langle x_t, x_t \rangle$

$= \langle w_t, x_t \rangle - 1$

Perceptron Example

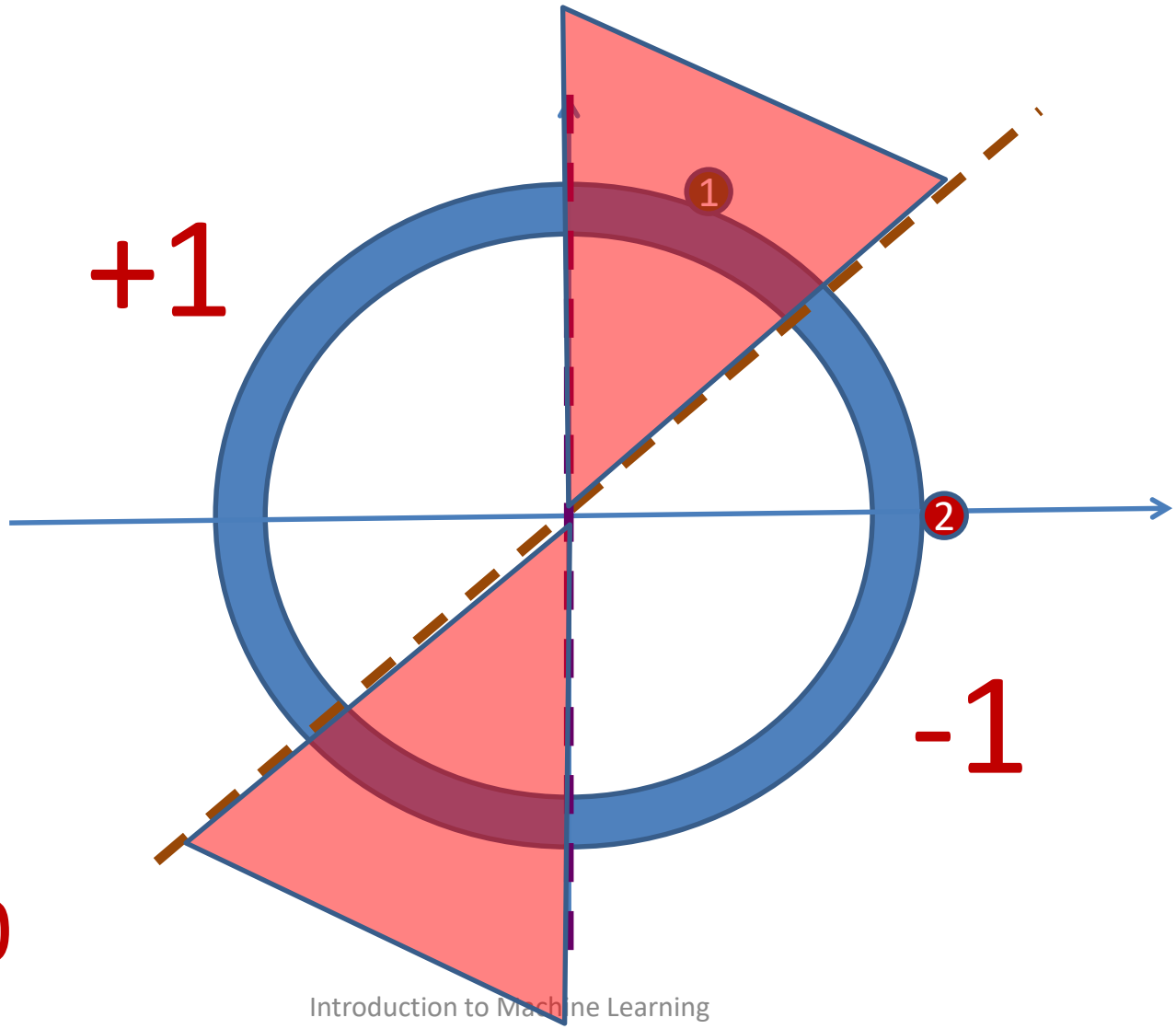
$w_1 = (0,0)$
$w_2 = (0,0)$



$x_1 - x_2 = 0$

Perceptron Example

$w_1 = (0,0)$
$w_2 = (0,0)$
$w_3 = (-1,0)$



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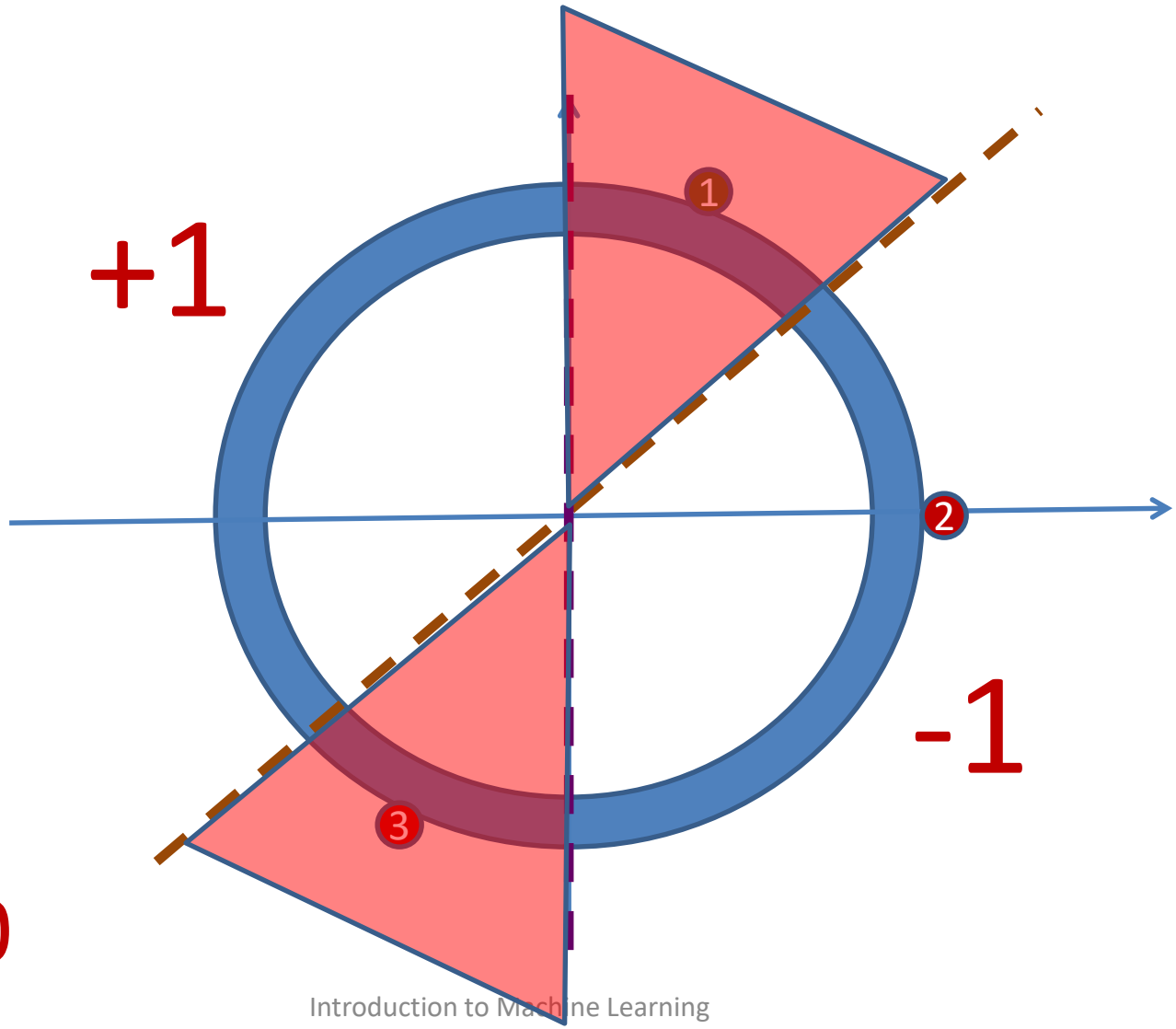
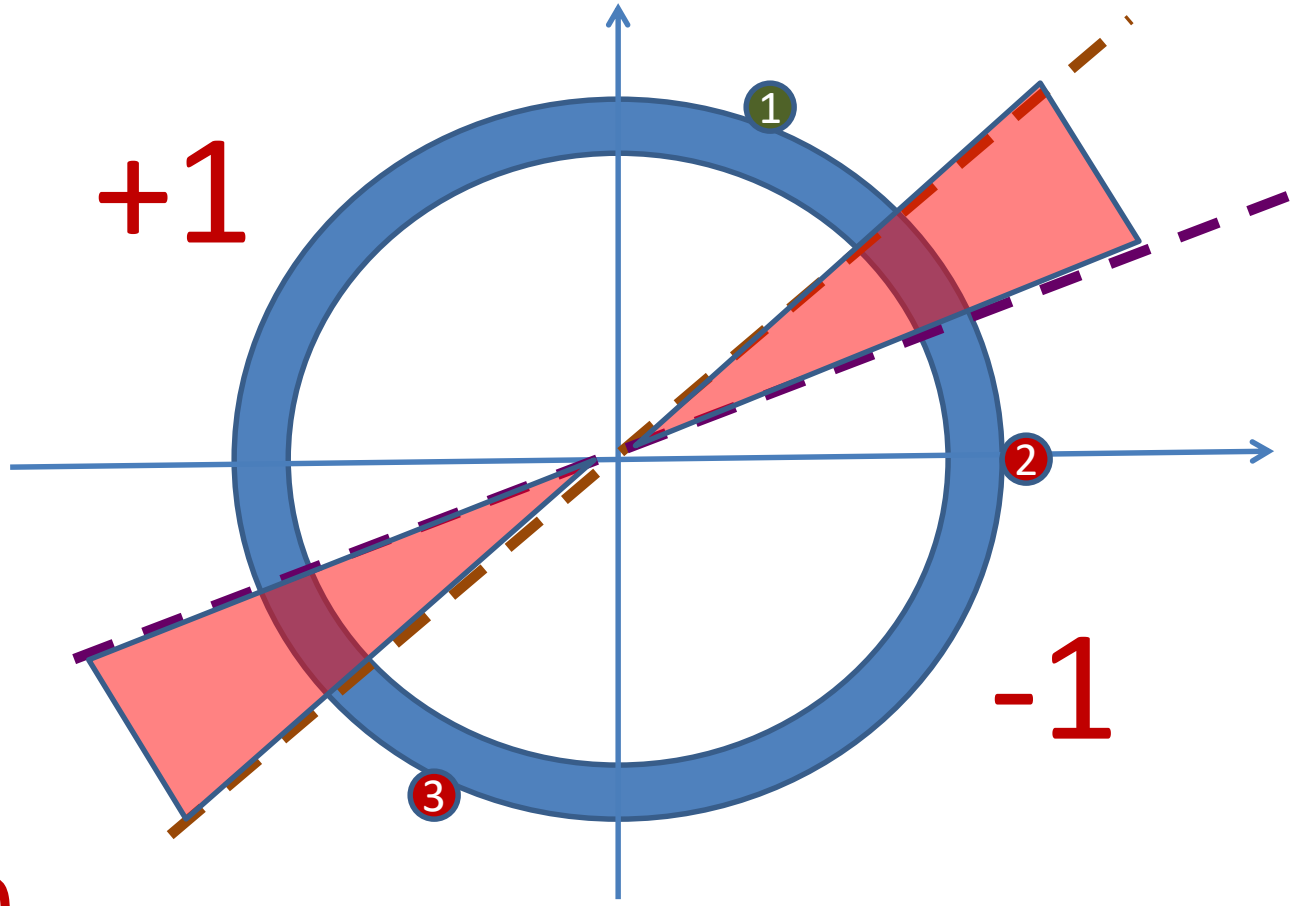


$x_1 - x_2 = 0$

Perceptron Example

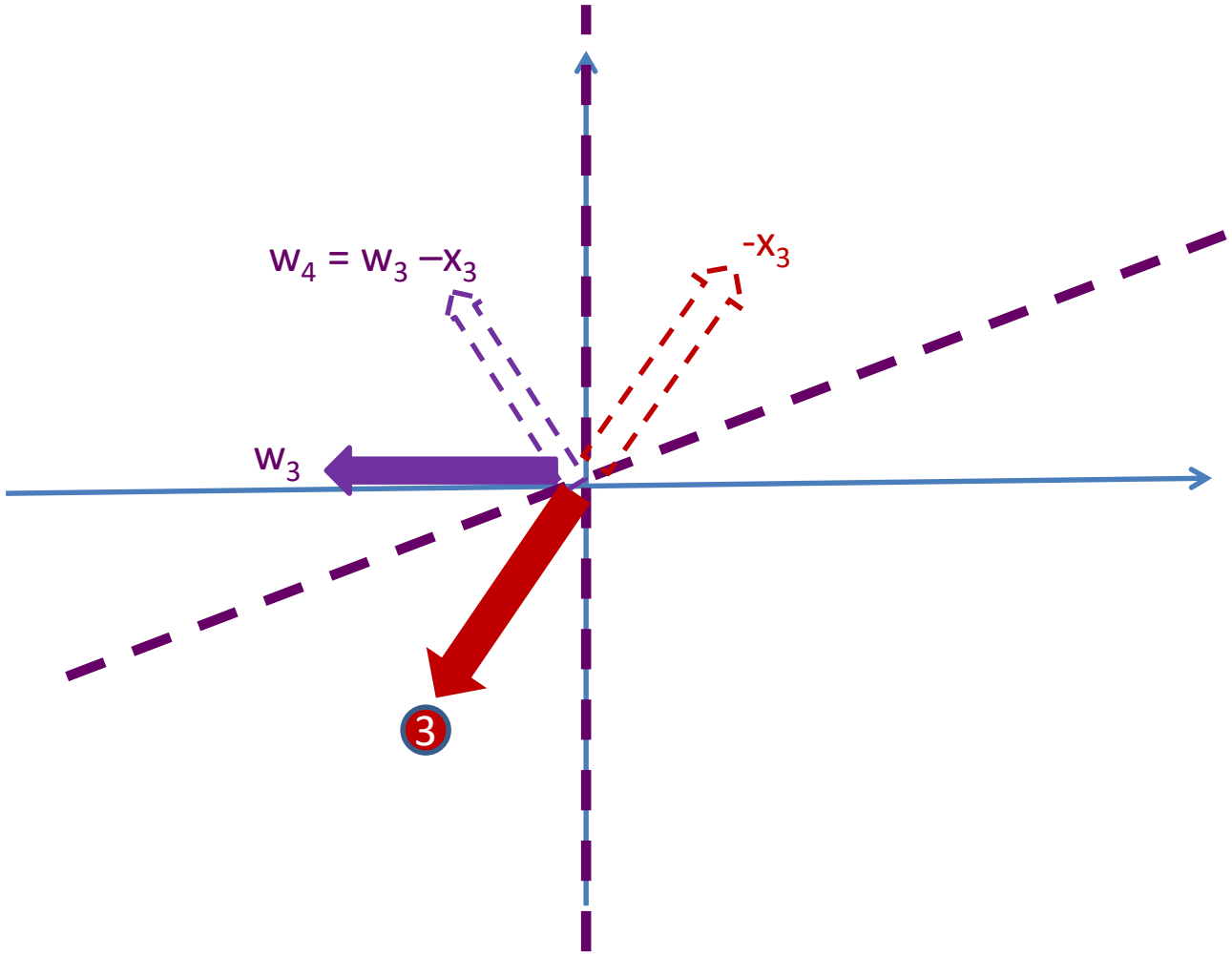
$w_1 = (0,0)$
$w_2 = (0,0)$
$w_3 = (-1,0)$
$w_4 = (-0.2,+0.6)$



$x_1 - x_2 = 0$

Perceptron - Geometric Interpretation

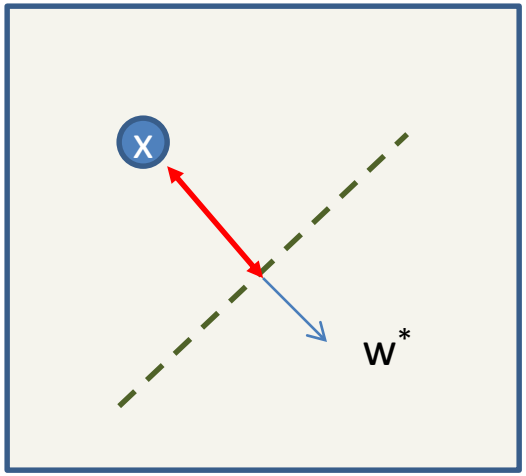
$w_1 = (0,0)$
$w_2 = (0,0)$
$w_3 = (-1,0)$
$w_4 = (-0.2,+0.6)$



Perceptron - Analysis

- target concept $c^*(x)$ uses w^* and $||w^*||=1$
- Margin γ:
  - For any *x is S*

$$\gamma = \min_{x \in S} \frac{|< x, w^* >|}{\|x\|}$$



- Theorem: *Number of mistakes ≤ $1/\gamma^2$*

Perceptron - Performance

Claim 1:
$\langle w_{t+1}, w^* \rangle \geq \langle w_t, w^* \rangle + \gamma$

Assume $c^*(x)=+1$

$\langle w_{t+1}, w^* \rangle =$

$\langle (w_t + x), w^* \rangle =$

$\langle w_t, w^* \rangle + \langle x, w^* \rangle \geq$

$\langle w_t, w^* \rangle + \gamma$

Similar for $c^*(x)=-1$

Claim 2: $||w_{t+1}||^2 \leq ||w_t||^2+1$

Assume $c^*(x)=+1$

$||w_{t+1}||^2 =$

$||w_t + x||^2 =$

$||w_t||^2 + 2\langle w_t, x \rangle + ||x||^2 \leq$

$||w_t||^2 + 1$

Since x is a mistake $\langle w_t, x \rangle$ is negative.

Similar for $c^*(x)=-1$

Perceptron - performance

Claim 3: $<w_t, w^*> \leq ||w_t||$

$$< w_t, w^* > \leq < w_t, \frac{w_t}{\|w_t\|} >= \|w_t\|$$

Completing the proof

- After M mistakes:

$<w_{M+1}, w^*> \geq \gamma M$ (claim1)

$||w_{M+1}||^2 \leq M$ (claim 2)

$$\gamma M \leq < w_{M+1}, w^* > \leq \|w_{M+1}\| \leq \sqrt{M}$$

$$M \leq \frac{1}{\gamma^2}$$

Perceptron

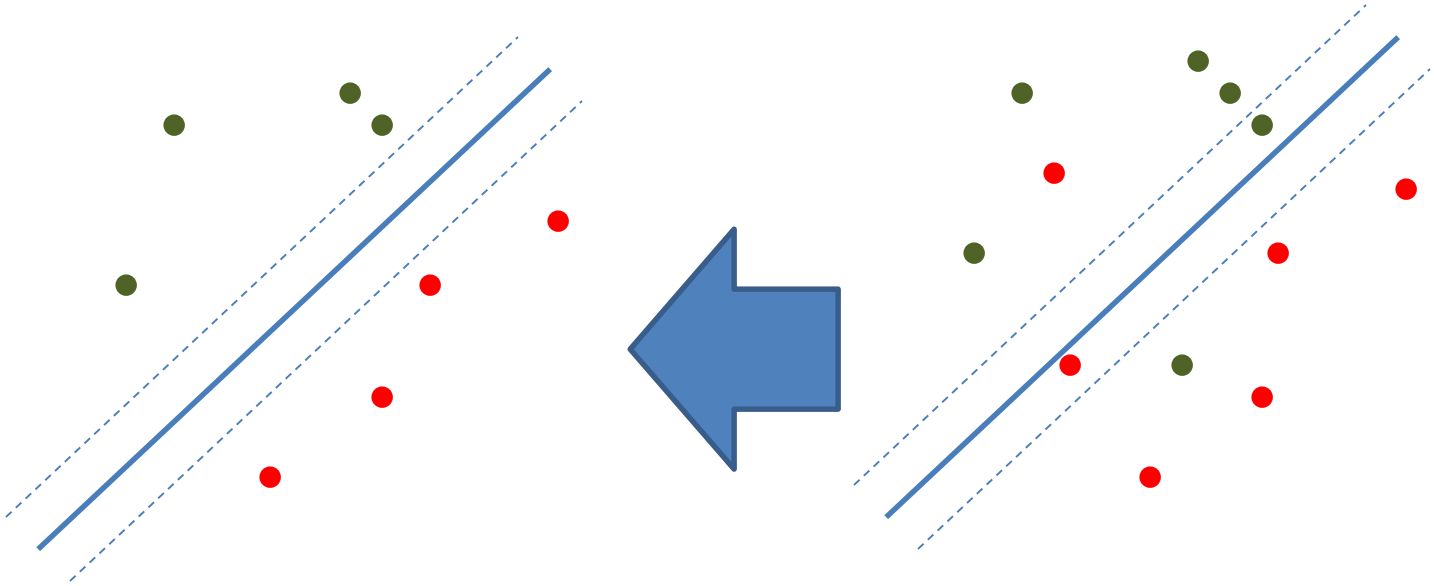
- Guaranteed convergence
  - realizable case
- Can be very slow (even for $\{0,1\}^d$)
- Additive increases:
  - problematic with large weights
- Still, a simple benchmark

Perceptron – Unrealizable case

Motivation

Realizable case

Unrelizable case



Hinge Loss

Motivation

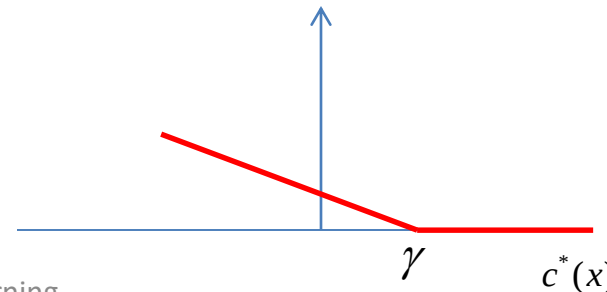
- "Move" points to be realizable
  - with margin γ
- correct points
  - both classification and margin
  - zero loss
- mistake points
  - even just margin
  - loss is the distance

Definition

- Assume $\langle x, w \rangle = \beta$
- Hinge Loss with margin γ:

$$\max\{0, 1 - \frac{c^*(x)\beta}{\gamma}\}$$

- Compare to
  Error: $c^*(x)\beta < 0$



Perceptron - Performance

- Let TD_γ=total distance

  $\Sigma_i \max\{0, \gamma - c^*(x)\beta_i\}$, where $\beta_i = \langle x_i, w^* \rangle$
- Claim 1': $\langle w_{M+1}, w^* \rangle \geq \gamma M - TD_\gamma$
- Claim 2: $||w_{t+1}||^2 \leq ||w_t||^2 + 1$
- Bounding the mistakes:

$$\sqrt{M} \geq \gamma M - TD_\gamma \quad \Rightarrow \quad M \leq \frac{1}{\gamma^2} + \frac{2}{\gamma} TD_\gamma$$

Winnow

Winnow –motivation

- Updates
  - multiplicative vs additive
- Domain
  - $\{0,1\}^d$ or $[0,1]^d$
    - we will use $\{0,1\}^d$
- Weights
  - non-negative
    - monotone function

- Separation
  - $c^*(x)=+1: \langle w^*,x\rangle \geq \theta$
  - $c^*(x)=-1: \langle w^*,x\rangle \leq \theta - \gamma$
  - $\theta \geq 1$
    - part of the input

- Remarks:
  - normalizing x in L_∞ to 1

Winnow - Algorithm

- parameter **$\beta > 1$**
  - we will use **$\beta = 1+\gamma/2$**
- Initialize **$w=(1, \dots, 1)$**
- predict **$h(x)=+1$** iff

  $\langle w,x \rangle \geq \theta$

- For a mistake:
- False Positive (**demotion**)
  - $c^*(x)=-1$, $h(x)=+1$
  - for every $x_i=1$: $w_i=w_i/\beta$
- False Negative (**promotion**)
  - $c^*(x)=+1$, $h(x)=-1$
  - for every $x_i=1$: $w_i=\beta w_i$

Winnow - intuition

- Demotion step
  - target negative
  - hypothesis positive
- Before update

  $\langle w,x \rangle = \alpha \geq \theta$
- After the update:

  $\langle w,x \rangle = \alpha/\beta < \alpha$
- Decrease in $\sum w_i$
  - at least $(1- \beta^{-1})\theta$

- Promotion step
  - target positive
  - hypothesis negative
- Before update

  $\langle w,x \rangle = \alpha < \theta$
- After the update:

  $\langle w,x \rangle = \alpha\beta > \alpha$
- Increase in $\sum w_i$
  - at most $(\beta-1)\theta$

Winnow - example

- Target function:
- $w^*=(2,2,0,0)$
- $\theta=2$, $\beta=2$

- What is the target function?
  - x_1 v x_2
  - monotone OR

- $w_0=(1,1,1,1)$
- $x_1=(0,0,1,1)$ $c_t(x_1)=-1$
  - $w_1=(1,1,½,½)$
- $x_2=(1,0,1,0)$ $c_t(x_2)=+1$
  - $w_2=(2,1,1,½)$
- $x_3=(0,1,0,1)$ $c_t(x_3)=+1$
  - $w_3=(2,2,1,1)$

Winnow - Theorem

- Theorem (realizable case)

Number of mistakes bounded by

$$O\left(\frac{1}{\gamma^2}\frac{d}{\theta}+\frac{\ln\theta}{\gamma^2}\sum_{i=1}^{d}w_i^*\right)$$

- Corollary: For θ=d we have $O\left(\frac{\ln d}{\gamma^2}\sum_{i=1}^{d}w_i^*\right)$

Winnow - Analysis

- Mistakes
  - u promotion steps
  - v demotion steps
  - mistakes = u+v
- Lemma 1:

due to sum equalities before, and since weights are positive

$$v \le \frac{\beta}{\beta - 1}\frac{d}{\theta} + \beta u$$

- Lemma 2: $w_i \le \beta\theta$

We won't promote unless wi by itself is not enough to pass theta

- Lemma 3:

  after u prom.

  and v demo.

  exists *i*

$$\log w_i \ge \frac{\theta u - (\theta - \gamma)v}{\sum_{i=1}^{d} w_i^*} \log \beta$$

- Proof of theorem

Winnow vs Perceptron

Percetron

- Additive updates
  - slow for large d
  - slow large weights
- Non-monotone
  - natural
- Simple Algorithm
- Margin scale $L_2(w^*)L_2(x)$

Winnow

- Multiplicative updates
  - handles large d nicely
  - ok with large weights
- Monotone
  - need to make monotone
  - flip non-monotone attributes
- Simple Algorithm
- Margin scale $L_1(w^*)L_\infty(x)$
- Additional factor log d
  - for $\theta=d$

Summary

Linear Separators

- Today: Perceptron and Winnow
- Next week: SVM
- 2 weeks: Kernels
- later: Adaboost

Brief history:

- Perceptron
  - Rosenblatt 1957
- Fell out of favor in 70s
  - representation issues
- Reemerged with Neural nets
  - late 80s early 90s
- Linear separators:
  - Adaboost and SVM
- The immediate future: deep learning